RELATIVISTIC QUANTUM MECHANICS

Tuesday 08-04-2014, 14.00-17.00

On the first sheet write your name, address and student number. Write your name on **all** other sheets. The total number of points is 90. You can earn 5 points for each subquestion, except for 4.3 and 4.4 - for each of these you can earn 15 points.
Use conventions with $\hbar = c = 1$. The chiral representation of the 4×4 gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

PROBLEM 1: LORENTZ TRANSFORMATIONS

1.1 How many generators does the Lorentz group have, and to which transformations do these correspond?

1.2 What are the generators of the Lorentz group for the scalar, vector and spinor representation?

1.3 Do two generators in the scalar representation generically commute? Do two generators in the vector representation generically commute? Does a generator in the vector representation generically commute with one in the spinor representation?

PROBLEM 2: HAMILTONIAN FORMALISM

The Lagrangian for a relativistic massive spinor field ψ is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \,. \tag{1}$$

2.1 What is the Euler-Lagrange equation for ψ?

2.2 What is the momentum conjugate to ψ, and what is the corresponding Hamiltonian density?

2.3 What are the Hamiltonian equations for this theory? Indicate the relations to the result of question 2.1.

PROBLEM 3: CHIRALITY

3.1 Explain the concept of chirality and the definition of positive and negative chirality spinors.

3.2 Is the notion of chirality consistent with Lorentz invariance? Explain your answer.

3.3 Is the notion of chirality consistent with the Dirac equation for a massive spinor? Explain your answer.

PROBLEM 4: CANONICAL QUANTIZATION

The Hamiltonian for a relativistic massive scalar field ϕ is given by

$$H = \int d^3x \left(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\partial_i \phi)^2 + \tfrac{1}{2}m^2\phi^2\right). \tag{2}$$

4.1 What is the momentum Π conjugate to the field ϕ?

4.2 Which commutation relations do we impose on the operators ϕ and Π in the Schrödinger picture? Indicate the dependence on space and time.

The decomposition into plane waves and ladder operators $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ reads

$$\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} [a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}]. \tag{3}$$

4.3 Derive the commutation relations between the ladder operators from your answer to 4.2.

4.4 Derive the form of the Hamiltonian in terms of the ladder operators, and interpret the different terms.

4.5 Indicate what would happen to the previous result when imposing anti-commutation relations instead of commutation relations. Interpret your result.